Scissors mode in a superfluid Fermi gas

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We evaluate the frequencies of scissors modes for density and concentration fluctuations in a vapour of fermionic atoms placed in two hyperfine levels inside a spherical harmonic trap. Both the superfluid and the normal state are considered, with inclusion of the interactions at the random-phase level. Two main results are obtained: (i) the transition to the superfluid state is signalled by the disappearance of soft transverse modes of the normal fluid in the collisionless regime and (ii) the eigenfrequency of the density fluctuations in the superfluid coincides with that of the normal fluid in the collisional regime. The latter property is related to the opening of the gap in the single-pair spectrum.

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I. INTRODUCTION

The possibility of obtaining a novel superfluid system by pairing of ultra-cold fermionic alkali atoms seems to become closer to experimental realization [1,2]. Various mechanisms of pairing have already been proposed [3–5], the simplest one being an *s*-wave pairing between atoms belonging to two different hyperfine levels of a magnetic trap.

Revealing the superfluid transition in a Fermi gas is expected to be more difficult than for bosonic condensation, since it cannot be inferred from the observation of the density profile of the cloud [4]. One has therefore to look for the transition in dynamical or kinematical properties of the fluid. We suggest in this work a possible method for the detection of the superfluid character of the transition. The same test has already been successfully employed in the case of a Bose-Einstein condensate [6,7].

The property of superfluidity in a vapour is related by definition to the character of its dynamical response to a long-wavelength transverse probe [8], which will excite only the non-superfluid component. In an inhomogeneous finite system, such as an atomic vapour in magnetic or optical confinement or an atomic nucleus, a test of the superfluidity can be obtained from the excitation of a small-angle oscillation in a plane where the confinement is slightly anisotropic. For a Bose-condensed gas this *scissors mode* is predicted to show only one frequency component in the superfluid state, while it has two frequency components in the normal fluid due to the additional contribution of transverse excitations. The study of the scissors mode in Bose-Einstein condensates of alkali vapours has been suggested by Guéry-Odelin and Stringari [6] and an experimental realization of their ideas has already been given by Maragò *et al.* [7].

The low-energy collective excitations of a trapped Fermi gas in the superfluid state have been investigated by Baranov and Petrov [9] in the dilute (non-interacting) limit. Their theory is formulated in terms of the fluctuations in the phase of the order parameter, but they show that these modes also manifest themselves as density fluctuations in the sample. Their equation of motion for the density fluctuations of a superfluid Fermi gas coincides with that obtained by Amoruso *et al.* [10] for a non-interacting normal Fermi fluid in the collisional regime, within a local-density approximation (LDA).

In this paper we derive in Sect. II an equation of motion for the density fluctuations of a superfluid Fermi gas in an improved LDA, which includes also the effects of the interactions within a random-phase approximation (RPA) as already proposed in early work by Anderson [11]. We make use of this equation in Sect. III to give theoretical predictions for the excitation frequency of the scissors mode in the superfluid Fermi gas. This result is compared in Sect. IV with the excitation frequencies of the scissors modes of a normal Fermi fluid in the degenerate regime and in the presence of interactions. Finally, Sect. V presents a summary and some concluding remarks.

II. COHERENT DYNAMICS OF A SUPERFLUID FERMI GAS

The low-energy excitation spectrum of a neutral Fermi gas with weak attractive forces at very low temperatures is characterized in the superfluid phase by a longitudinal collective mode with a linear dispersion relation at the velocity of ordinary (first) sound. This mode is not allowed to propagate in the normal phase as it is embedded in the particle-hole continuum of excitations (just as it is the case for the zero-sound mode in a fluid with attractive interactions).

However, in the superfluid phase the opening of the gap Δ shifts the energy threshold for the quasiparticle-quasihole continuum to 2Δ, leaving a window in energy where first sound is stable.This mode is known as the Bogolubov-Anderson sound [11,12] and can be viewed as a coherent oscillation of the order parameter for the condensed phase; in fact, its presence is required by the Goldstone theorem to ensure gauge invariance. The simplest theory which includes the Bogolubov-Anderson mode in the density fluctuation spectrum is the random-phase approximation for a superconductor [11,13,14].

In a harmonically trapped superfluid the Bogolubov-Anderson sound becomes quantized due to the confinement. In the dilute limit the equation of motion for the density fluctuations has been obtained by Baranov and Petrov [9] using a semiclassical solution of the Bogolubov - de Gennes equations. As already noted in Sect. I, the resulting spectrum of excitations coincides with that obtained [10] for an harmonically trapped normal Fermi gas in the collisional regime, when a local-density approximation for the kinetic stress tensor is employed.

The coincidence of the two spectra can be explained – within the LDA – by making use of the result for the homogeneous fluid that the compressibility is unaffected by the phase transition [15]: a superfluid and a normal fluid at the same density and interaction strength show the same value for the compressibility. Indeed this property is not restricted to the non-interacting Fermi gas: the coincidence of the sound velocity in a superfluid and in a normal fluid in the collisional regime has also been demonstrated by Larkin and Migdal [16] for generic interaction strengths within the theory of superfluid Fermi liquids. For the specific case of a weakly interacting Fermi gas the result for the sound velocity c in the normal fluid in the collisional regime, given by $c^2 = (v_f^2/3)(1 + 2k_f a_{\uparrow\downarrow}/\pi)$ coincides with the velocity of the Bogolubov-Anderson mode, as obtained from the RPA [17]. Here, v_f and k_f are the Fermi velocity and wavenumber, while $a_{\uparrow\downarrow}$ is the s-wave scattering length between fermions in different hyperfine states.

Another characteristic of the spectrum of a homogeneous Fermi superfluid is that all collective modes other than the density fluctuations, such as spin-density and transverse current-density modes, are suppressed in the zero temperature limit, since they are related to the motion of the normal component of the fluid [18,19].

These observations allow us to conclude that the equations of motion for the total density fluctuations in a symmetric two-component normal Fermi gas in the collisional regime [20] describe as well the collective excitations of a weakly interacting superfluid at $T = 0$ within a local-density approximation. Specifically, we take as the basic equations for the dynamics of a superfluid Fermi gas in the linear regime the following:

$$\partial_t n(\mathbf{r}, t) = -\nabla \cdot (n_{eq}(\mathbf{r})\mathbf{v}(\mathbf{r}, t)) \tag{1}$$

and

$$\partial_t \mathbf{v}(\mathbf{r}, t) = -\nabla \left[\frac{1}{3} A (n_{eq}(\mathbf{r})/2)^{-1/3} n(\mathbf{r}, t) + \frac{1}{2} g n(\mathbf{r}, t) \right] . \tag{2}$$

Here $A = \hbar^2(6\pi^2)^{2/3}/2m$, $g = 4\pi\hbar^2 a_{\uparrow\downarrow}/m$, and $n(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ are the total density fluctuation and velocity field for the gas, $n_{eq}(\mathbf{r})$ being the equilibrium density profile. Interactions have been included in the theory at mean field (Hartree) level and exchange is not allowed since s-wave interactions are active only between fermions with different spin polarization. In the homogeneous fluid this approximation on the interactions leads to the RPA expression for the sound velocity [17]. The results of Baranov and Petrov [9] are recovered by setting $g = 0$ in Eq. (2).

It is interesting to notice that Eq. (2) for the velocity field is irrotational, in analogy with the corresponding equation for a Bose-condensed gas [21]. In this case it contains explicitly the effect of Fermi statistics in the expression for the linearized local chemical potential. Equations (1) and (2) apply only for describing low-energy excitations at very low temperature, since they do not include the coupling with thermal quasiparticle excitations and the related damping.

As an application of Eqs. (1-2) we study the possibility of exciting the scissors mode in a superfluid Fermi gas.

III. SCISSORS MODE IN A SUPERFLUID FERMI GAS

We consider an atomic cloud confined inside a slightly anisotropic trap in the xy plane, described by the confining potential

$$V_{ext}(\mathbf{r}) = \frac{1}{2} m\omega_0^2 (1 + \varepsilon) x^2 + \frac{1}{2} m\omega_0^2 (1 - \varepsilon) y^2 + \frac{1}{2} m\omega_z^2 z^2 . \tag{3}$$

Within this geometry the scissors mode can be excited by a sudden rotation of the trap through a small angle in the xy plane. In the experiment by Maragò *et al.* [7] the anisotropy has been obtained by adding a small component in the z direction to the magnetic field of the TOP trap and the scissors mode has been excited by a sudden change of the sign of such field.

For the study of the scissors mode of a superfluid Fermi gas we employ the technique of moments developed in [22] (see also [6]). We define the dynamical average of a variable $\chi(\mathbf{r},\mathbf{v})$ on the total density profile $n(\mathbf{r},t)$ as

$$\langle\chi\rangle = \int d^3r\,\chi(\mathbf{r},\mathbf{v})n(\mathbf{r},t)\ . \tag{4}$$

For the scissors mode the relevant variable is $\chi = xy$, describing the quadrupolar oscillation with z-component of the angular momentum $m_z = \pm 2$. By making use of the equations of motion for the superfluid we obtain

$$\partial_t\langle xy\rangle = \langle xv_y + yv_x\rangle \tag{5}$$

from the continuity equation and

$$\partial_t\langle xv_y + yv_x\rangle = -2\omega_0^2\langle xy\rangle \tag{6}$$

from the linearized Euler equation. A physical picture of the scissors mode can be obtained by observing that the moment $\langle xy\rangle$ is related to the angle θ of oscillation of the cloud in the xy plane, if $\theta \ll 1$. For larger angles instead the variable $\langle xy\rangle$ describes the usual quadrupolar excitation of the cloud.

In deriving Eq. (6) we have made use of the expression for the equilibrium density profile in the local-density ("Thomas-Fermi") approximation [23,24], which for a symmetric two-component fluid is the solution of $A[n_{eq}(\mathbf{r})/2]^{2/3} = \mu - V_{ext}(\mathbf{r}) + gn_{eq}(\mathbf{r})/2$. We are here exploiting the fact that even in the presence of weak interactions the equilibrium density profile of the superfluid is well approximated by the corresponding profile of a normal fluid at the same number of particles. This property is illustrated in Figure 1 and is a consequence of the interplay between the external confinement and interactions. While interactions shift only slightly the chemical potential of the homogeneous superfluid relative to that of the normal fluid, in the presence of confinement they also enter in the density profiles through the position-dependent Hartree term. This second effect is dominant in determining the profiles, as shown in Figure 1. As for confined Bose condensates, the fluid is dilute but nevertheless interactions significantly modify the density profile.

The solution of the coupled equations (5) and (6) yields for the frequency of the mode the result

$$\omega^2 = 2\omega_0^2\ . \tag{7}$$

This value coincides with that for the $(n = 0\ l = 2)$ total-density surface mode of a two-component normal Fermi gas in the collisional regime [20]. The same property holds for a Bose-condensed cloud [6].

In fact, the scissors mode is not affected by the interactions: its frequency coincides with that of the $(n = 0\ l = 2)$ surface mode in a single-component Fermi gas [9,10]. This is indeed a property of all surface modes of total density fluctuations [20]. For these modes $\nabla\cdot\mathbf{v} = 0$ and the equation of motion for the velocity field reduces to

$$m\partial_t^2\mathbf{v}(\mathbf{r},t) = \nabla\left[\mathbf{v}(\mathbf{r},t)\cdot\nabla[A(n_{eq}(\mathbf{r})/2)^{2/3} + gn_{eq}(\mathbf{r})/2]\right]\ . \tag{8}$$

By employing the Thomas-Fermi form of the equilibrium profile we have

$$m\partial_t^2\mathbf{v}(\mathbf{r},t) = -\nabla\left[\mathbf{v}(\mathbf{r},t)\cdot\nabla V_{ext}(\mathbf{r})\right]\ . \tag{9}$$

This equation shows that only the external confinement determines the frequency of these modes. Equation (9) is independent of the statistics: it describes as well the surface modes of a classical gas in the hydrodynamic regime and of a Bose-Einstein condensate at $T = 0$ [25].

IV. SCISSORS MODES FOR A NORMAL FERMI GAS

We shall now contrast the result (7) for the superfluid with the excitation frequencies of the scissors modes in a two-component Fermi fluid in the normal state. At a temperature higher than the superfluid transition temperature T_{sup} the gas is described by the Vlasov-Landau equation for the Wigner distribution functions $f_{\mathbf{p}}^{\sigma}(\mathbf{r},t)$ of each of the components:

$$\left(\partial_t + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}}(V_{ext}(\mathbf{r}) + gn_{\bar\sigma}(\mathbf{r},t))\cdot\nabla_{\mathbf{p}}\right)f_{\mathbf{p}}^{\sigma}(\mathbf{r},t) = I_{coll}[f_{\mathbf{p}}^{\sigma}(\mathbf{r},t)]\ . \tag{10}$$

The density profile of each component is defined in terms of the Wigner function as $n_\sigma(\mathbf{r},t) = \int d^3p f_{\mathbf{p}}^{\sigma}(\mathbf{r},t)/(2\pi\hbar)^3$ and I_{coll} is the collision integral.

The average of the variable $\chi(\mathbf{r},\mathbf{v})$ on each component σ of the gas is defined as

$$\langle\chi\rangle_\sigma = \frac{1}{(2\pi\hbar)^3}\int d^3r \int d^3p\, \chi(\mathbf{r},\mathbf{v}) f^\sigma_{\mathbf{p}}(\mathbf{r},t) \ . \tag{11}$$

The equation of motion for $\langle xy\rangle_\sigma$ is now coupled to both the longitudinal and the transverse excitations and to the kinetic tensor fluctuations, yielding the following set of equations:

$$\partial_t\langle xy\rangle_\sigma = \langle xv_y + yv_x\rangle_\sigma \ , \tag{12}$$

$$\partial_t\langle xv_y - yv_x\rangle_\sigma = -2\varepsilon\omega_0^2(C\langle xy\rangle_\sigma + (1-C)\langle xy\rangle_{\bar\sigma}) \ , \tag{13}$$

$$\partial_t\langle xv_y + yv_x\rangle_\sigma = 2\langle\Pi_{xy}\rangle_\sigma/m - 2\omega_0^2(C\langle xy\rangle_\sigma + (1-C)\langle xy\rangle_{\bar\sigma}) \tag{14}$$

and

$$\partial_t\langle\Pi_{xy}\rangle_\sigma = -mC\omega_0^2\left[\langle xv_y + yv_x\rangle_\sigma + \varepsilon\langle xv_y - yv_x\rangle_\sigma\right] - \langle\Pi_{xy}\rangle_\sigma/\tau \ . \tag{15}$$

We have defined the current density $n_\sigma\mathbf{v}_\sigma = \int d^3p\, \mathbf{p} f^\sigma_{\mathbf{p}}(\mathbf{r},t)/m(2\pi\hbar)^3$ and the xy component of the kinetic stress tensor $\Pi^\sigma_{xy} = \int d^3p\, p_x p_y f^\sigma_{\mathbf{p}}(\mathbf{r},t)/m(2\pi\hbar)^3$, and made use of the Thomas-Fermi Ansatz $An^\sigma_{eq}(\mathbf{r})^{2/3} = C(E_F - V_{ext}(\mathbf{r}))$ for the equilibrium density profile in a symmetric system, as already employed in [20]. The constant $C = (E_F^0/E_F)^2$, where E_F is the true Fermi energy and E_F^0 is that of the non-interacting Fermi gas, measures the strength of the interactions in its deviations from unity. The approximations that we have made require $|C-1| \ll 1$ and $T \ll T_F$; the latter condition is well compatible with the condition $T > T_{sup}$ in a weakly interacting fluid. Notice that the motions of the two components of the gas are coupled for $C \neq 1$.

Collisions have been included in Eq. (15) within a single-relaxation-time approximation by setting

$$\langle\Pi_{xy} I_{coll}\rangle_\sigma = -\langle\Pi_{xy}\rangle_\sigma/\tau \ . \tag{16}$$

This approximation subsumes damping by scattering against impurities and by inter-species scattering (see [20] for a full discussion).

In the collisionless regime ($\omega\tau \gg 1$), equations (12-15) can be combined to yield the following coupled differential equations for $\langle xy\rangle_\sigma$:

$$(\partial_t^4 + 4C\omega_0^2\partial_t^2 + 4\varepsilon^2\omega_0^4C^2)\langle xy\rangle_\sigma + 2\omega_0^2(1-C)(-\partial_t^2 + 2\varepsilon^2\omega_0^2)\langle xy\rangle_{\bar\sigma} = 0 \ . \tag{17}$$

Solution by diagonalization yields two scissors modes associated with total density fluctuations and two further modes associated with concentration fluctuations, at frequencies given by

$$\omega^2 = 2\omega_0^2\begin{cases} 2C \mp (1-C) \\ \varepsilon^2 C(C \pm (1-C))/(2C \mp (1-C)) \end{cases} \tag{18}$$

The appearance of soft modes with a frequency proportional to ε is a peculiarity of a non-superfluid system.

In the collisional regime ($\omega\tau \ll 1$), Eqs. (12-15) instead yield

$$(\partial_t^3 + 2C\omega_0^2\partial_t)\langle xy\rangle_\sigma + 2(1-C)\omega_0^2\partial_t\langle xy\rangle_{\bar\sigma} = 0 \ . \tag{19}$$

This has two solutions for non-zero frequencies of the scissor modes at

$$\omega^2 = 2\omega_0^2(C \pm (1-C)) \ . \tag{20}$$

These agree with the ($n = 0$ $l = 2$) surface modes already obtained by a different approach in [20]. As expected, the frequency of the total-density mode reproduces the result in Eq. (7).

V. SUMMARY AND CONCLUDING REMARKS

In this work we have extended the equations of motion for density fluctuations in an inhomogeneous Fermi gas in the superfluid state at $T = 0$ to include interactions at a mean-field (RPA) level within a local-density approximation. In the appropriate limits our approach reproduces Anderson result [11,17] for the speed of the Bogolubov-Anderson phonon in the homogeneous superfluid and those by Baranov and Petrov [9] for a confined non-interacting superfluid.

As an application, we have computed the frequency of the scissors mode for a fermionic superfluid gas. By comparing the result with those for a normal Fermi gas in the collisionless regime, we have shown that the measurement of the scissors modes could be used as a signal for the superfluid transition, since well below the transition temperature the soft transverse modes are suppressed. However, these modes are already non-propagating in a normal Fermi fluid in the collisional regime.

We have also found that the excitation frequency of the scissors mode in a superfluid Fermi gas coincides with that of a normal Fermi gas in the collisional regime. This result is understood in the fermionic case as related to the opening of the gap which stabilizes the first-sound mode. As is the case for all surface modes, interactions do not shift the frequency of the scissors mode in the superfluid.

While we have worked here within a local-density approximation, further work should be addressed to treat in a fully quantal way the dynamical properties of the inhomogeneous fluid of present interest.

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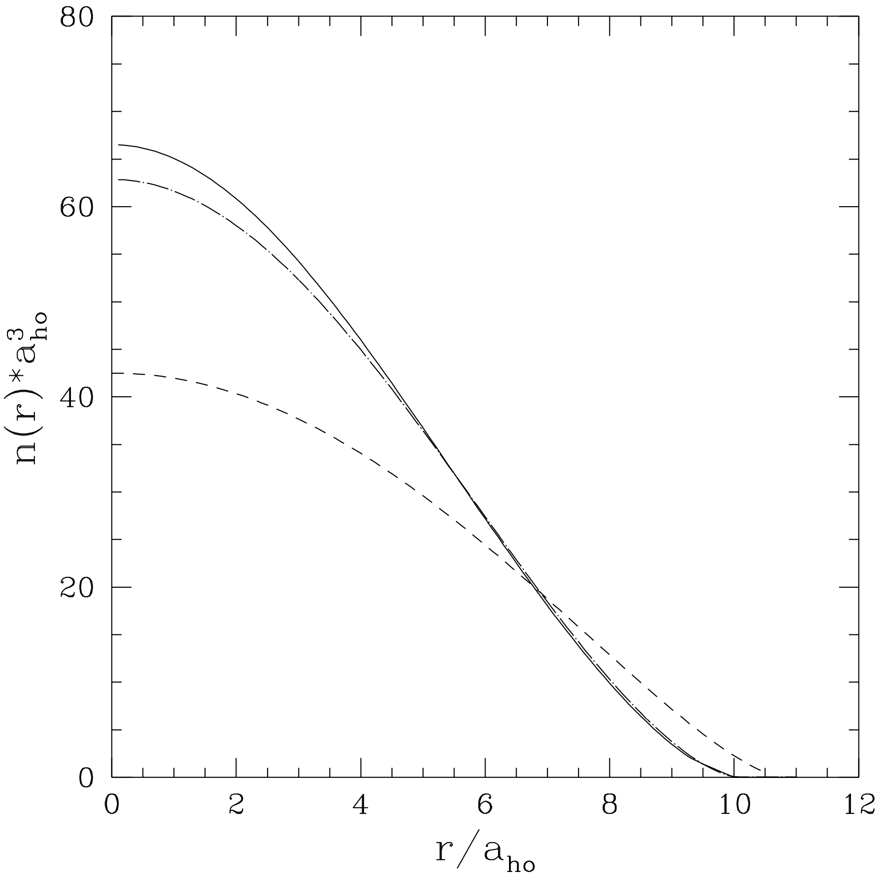

FIG. 1. Equilibrium density profile at $T = 0$ in spherical harmonic confinement (with trapping frequency $\omega_{ho} = 2\pi \times 520$ s^{-1}) for a superfluid Fermi gas (solid line) and for a normal Fermi gas (dot-dashed line) in the symmetric system at the same number of particles $N = 6.6 \times 10^4$. The two curves are the result of a local-density approximation using the chemical potential of the homogeneous fluid, as evaluated in the first case from the solution of the BCS equations, and in the second from the non-interacting value $\mu_{hom}[n] = \hbar^2(3\pi^2 n)^{2/3}/2m$. The equilibrium density profile of a non-interacting normal Fermi gas at the same N is also shown (dashed line).